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An Improved Cosmological Model

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ABSTRACT

We study a class of non-local, action-based, and purely gravitational models. These models seek to describe a cosmology in which inflation is driven by a large, bare cosmological constant that is screened by the self-gravitation between the soft gravitons that inflation rips from the vacuum. Inflation ends with the universe poised on the verge of gravitational collapse, in an oscillating phase of expansion and contraction that should lead to rapid reheating when matter is included. After the attainment of a hot, dense universe the nonlocal screening terms become constant as the universe evolves through a conventional phase of radiation domination. The onset of matter domination triggers a much smaller anti-screening effect that could explain the current phase of acceleration.

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1 Introduction

On scales larger than about $100Mpc$ the universe is well described by the geometry:

$$ds^2 = -dt^2 + a^2(t) d\mathbf{x} \cdot d\mathbf{x} . \quad (1)$$

The time variation of the scale factor $a(t)$ gives the instantaneous values of the Hubble parameter $H(t)$ and the deceleration parameter $q(t)$ or, equivalently, the first slow-roll parameter ϵ :

$$H(t) \equiv \frac{\dot{a}(t)}{a(t)} = \frac{d}{dt} \ln a(t) , \quad (2)$$

$$q(t) \equiv -\frac{\dot{a}(t) \ddot{a}(t)}{\dot{a}^2(t)} = -1 - \frac{\dot{H}(t)}{H^2(t)} \equiv -1 + \epsilon(t) . \quad (3)$$

Their current values are: $H_0 \simeq (67.8 \pm 0.9) km/sec Mpc$ and $\epsilon_0 \simeq 0.462 \pm 0.017$ [1].¹

There is overwhelming evidence that the history of the universe included a period of very early ($t \sim 10^{-33} sec$) accelerated expansion known as inflation and defined by $H > 0$ with $\epsilon < 1$ [3, 4]. During the inflationary era infrared gravitons are produced out of the vacuum because of the accelerated expansion of spacetime [5]. The interaction stress among the gravitons produced – an inherently non-local effect – can lead to a non-trivial quantum gravitational back-reaction on inflation [6]. General counting rules give the following leading infrared behaviour for the Hubble parameter $H(t)$ at late times in de Sitter spacetime [7]:

$$H(t) = H_{in} \left\{ 1 - G\Lambda \left(c_2 G\Lambda \ln[a(t)] + c_3 (G\Lambda)^2 \ln^2[a(t)] + \dots \right) \right\} . \quad (4)$$

It becomes evident from (4) that perturbation theory breaks down when $\ln[a(t)] \sim (G\Lambda)^{-1}$ and that evolving beyond this point requires non-perturbative techniques.

In the absence of non-perturbative results, it is perhaps desirable to propose phenomenological models that can provide calculable evolution beyond perturbation theory [8]. This can be accomplished by modifying the field equations:

$$G_{\mu\nu} + \Delta G_{\mu\nu}[g] = -\Lambda g_{\mu\nu} , \quad (5)$$

¹In quoting these numbers we have used fits from cosmic ray microwave data which effectively exploits the Λ CDM model for $z \sim 1000$. Larger and significantly different values for H_0 arise from Hubble plots which exploit Λ CDM for $z \sim 1$ [2].

where $\Delta G_{\mu\nu}[g]$ encodes the full effect of the quantum-induced gravitational back-reaction.²

Any such model should:

- Be consistent with the perturbative expectation (4).
- Reflect the non-local nature of the back-reaction effect in a causal way.
- Respect stress-energy conservation.
- Not disturb the basic ability of the gravitational equations to evolve from the initial spacelike surface with knowledge of the metric and its first time derivative only.

The hope is that the actual construction of the model will contain the most cosmologically significant part of the full effective quantum gravitational equations.

Previously [9, 10] we proposed a phenomenological model based on an effective conserved stress-energy tensor $T_{\mu\nu}[g]$:

$$\Delta G_{\mu\nu}[g] = -8\pi G T_{\mu\nu}[g] \quad , \quad (6)$$

which takes the perfect fluid form:

$$T_{\mu\nu}[g] = (\rho[g] + p[g])u_\mu[g]u_\nu[g] + p[g]g_{\mu\nu} \quad , \quad g^{\mu\nu}u_\mu[g]u_\nu[g] = -1 \quad , \quad (7)$$

where the ansatz for the pressure $p[g]$ is:

$$p[g] = \Lambda^2 f(Y) \quad , \quad Y \equiv -G\Lambda \frac{1}{\Box} R \quad . \quad (8)$$

It can be shown [9] that all models of this generic type where the function $f(Y)$ grows monotonically and without bound:

- Experience a long phase of inflation.
- The end of inflation leads to a short phase of oscillations.
- The participation of all super-horizon modes to the oscillations furnishes a natural and very fast reheating mechanism for the cosmos using only the universal gravitational coupling to matter.
- If matter couplings that allow energy dissipation are added it is plausible

²Hellenic indices take on spacetime values while Latin indices take on space values. Our metric tensor $g_{\mu\nu}$ has spacelike signature and our curvature tensor equals: $R^\alpha_{\beta\mu\nu} \equiv \Gamma^\alpha_{\nu\beta,\mu} + \Gamma^\alpha_{\mu\rho}\Gamma^\rho_{\nu\beta} - (\mu \leftrightarrow \nu)$. The initial Hubble constant is $3H_{\text{in}}^2 \equiv \Lambda$. We restrict our analysis to scales $M \equiv (\Lambda/8\pi G)^{\frac{1}{4}}$ below the Planck mass $M_{\text{Pl}} \equiv G^{-\frac{1}{2}}$ so that the dimensionless coupling constant $G\Lambda$ of the theory is small.

that the epoch of oscillations ends in a radiation domination epoch.

However, these models have negative attributes as well:

- There is a “sign problem” because their post-inflationary evolution eventually makes the pressure positive and thus in conflict with the observed late time acceleration [11, 12].
- There is a “magnitude problem” because the magnitude of the total pressure produced by the source is unacceptably large relative to the current pressure.

There is another generic class of models where the source of $\Delta G_{\mu\nu}[g]$ is a quantum-induced non-local effective action term $\Delta S[g]$:

$$\Delta G_{\mu\nu}[g] = \frac{16\pi G}{\sqrt{-g}} \frac{\delta \Delta S[g]}{\delta g^{\mu\nu}} \quad , \quad \Delta S \equiv \int d^4x \Delta \mathcal{L}[g] \quad , \quad (9)$$

where we parametrize $\Delta \mathcal{L}[g]$ as follows:

$$\Delta \mathcal{L}[g] = \Lambda^2 h(X[g]) \sqrt{-g} \quad . \quad (10)$$

The purpose of this paper is to present a phenomenological model of the latter kind that does not suffer from the sign and magnitude problems described above. Section 2 describes the construction of the model and derives the dynamical equations it satisfies. Section 3 presents numerical and semi-analytical results for the cosmology predicted. Our conclusions comprise Section 4.

2 The Model

Simple tools to construct a reasonable *ansatz* for the most cosmologically significant part of the full effective action are:

- Curvature invariants whose specialization to the geometry (1) gives:

$$R = 6(2 - \epsilon)H^2 \quad , \quad R^2 = 36(2 - \epsilon)^2 H^4 \quad , \quad (11)$$

$$R_{\mu\nu} R^{\mu\nu} = 12(3 - 3\epsilon + \epsilon^2)H^4 \quad , \quad (12)$$

$$R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} = 12(2 - 2\epsilon + \epsilon^2)H^4 \quad , \quad (13)$$

- Invariant differential operators whose inverses can plausibly introduce non-locality and whose specialization to the geometry (1) gives:

$$\square = \frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g} g^{\mu\nu} \partial_\nu \right) = -\partial_t^2 - 3H \partial_t \quad , \quad (14)$$

$$\square_c = \square - \frac{1}{6} R = -\partial_t^2 - 3H \partial_t - 2H^2 - \dot{H} \quad , \quad (15)$$

when acting on a function of co-moving time. Their inverses:

$$\frac{1}{\square} = \int_{t_{\text{in}}}^t dt' \frac{1}{a^3(t')} \int_{t_{\text{in}}}^{t'} dt'' a^3(t'') , \quad (16)$$

$$\frac{1}{\square_c} = \frac{1}{a(t)} \int_{t_{\text{in}}}^t dt' \frac{1}{a(t')} \int_{t_{\text{in}}}^{t'} dt'' a^2(t'') , \quad (17)$$

are defined with retarded boundary conditions to avoid the appearance of new degrees of freedom [13].

A way of achieving the desired properties for the induced source $X[g]$ in (10) is as follows:

- To address the “magnitude problem”, we must move the very high scale factor of $\Lambda = 3H_{\text{in}}^2$ which appears in (8) to the right and have it re-appear essentially as $H^2(t)$, a factor that decreases in magnitude like t^{-2} after inflation:

$$Y = -3GH_{\text{in}}^2 \frac{1}{\square} R \rightarrow -3G \frac{1}{\square} H^2 R . \quad (18)$$

An immediate consequence is that our ansatz now requires an extra curvature invariant – see (11-13) – to account for the extra factor of H^2 .

- To address the “sign problem”, $X[g]$ must change sign as we exit the inflationary epoch ($\epsilon < 1$) and enter the post-inflationary epoch ($\epsilon > 1$). By inspecting expressions (11-13) we conclude that the combination:

$$\frac{1}{3}R^2 - R_{\mu\nu}R^{\mu\nu} = 12(1 - \epsilon)H^4 , \quad (19)$$

indeed changes sign as ϵ passes through 1.

- The requirement for the effect to become quiescent during radiation domination ($\epsilon = 2$) is most easily satisfied by having the curvature scalar R present in the ansatz.
- The ansatz must contain an *overall* factor of \square^{-1} to account for the secular nature of the effect which implies the need for an operator with memory for the behaviour of the source by not extinguishing its effect as the universe evolves. Furthermore, \square^{-1} provides the single infrared logarithm dictated by the de Sitter correspondence limit (4) to order $(G\Lambda)^2$.
- The dimensionality of the ansatz for $X[g]$ requires the presence of a second inverse differential operator. To preserve the correspondence limit (4) this operator must not give an additional infrared logarithm to order $(G\Lambda)^2$ and it is \square_c^{-1} that has this property.

Therefore, the proposed *ansatz* consists of the following quantum-induced source $X[g]$:³

$$X[g] \equiv G \frac{1}{\square} R \frac{1}{\square_c} \left(\frac{1}{3} R^2 - R_{\mu\nu} R^{\mu\nu} \right) . \quad (20)$$

The function $h(X[g])$ must have the ability to end inflation which implies that it must have the ability to become singular.⁴ The contribution $\Delta G_{\mu\nu}$ to the gravitational field equations (5) is quite complicated and is most easily derived by going to the equivalent scalar representation [14, 15]. We introduce two auxiliary scalar fields A and C which we require to obey the following equations of motion:

$$\square_c A = \frac{1}{3} R^2 - R_{\mu\nu} R^{\mu\nu} , \quad (21)$$

$$\square C = R A . \quad (22)$$

This can be achieved by introducing two Lagrange multipliers B and D this way:

$$\begin{aligned} \Delta \mathcal{L} = & \Lambda^2 h(GC) \sqrt{-g} + B \left[\square_c A - \left(\frac{1}{3} R^2 - R_{\mu\nu} R^{\mu\nu} \right) \right] \sqrt{-g} \\ & + D [\square C - R A] \sqrt{-g} , \end{aligned} \quad (23)$$

so that the desired equations of motion (21-22) emerge:

$$\frac{1}{\sqrt{-g}} \frac{\delta(\Delta S)}{\delta B} = \square_c A - \left(\frac{1}{3} R^2 - R_{\mu\nu} R^{\mu\nu} \right) = 0 , \quad (24)$$

$$\frac{1}{\sqrt{-g}} \frac{\delta(\Delta S)}{\delta D} = \square C - R A = 0 , \quad (25)$$

as well as those for the Lagrange multipliers:

$$\frac{1}{\sqrt{-g}} \frac{\delta(\Delta S)}{\delta A} = \square_c B - R D = 0 \quad \Rightarrow \quad B = \frac{1}{\square_c} R D , \quad (26)$$

$$\frac{1}{\sqrt{-g}} \frac{\delta(\Delta S)}{\delta C} = \square D + G \Lambda^2 h'(GC) = 0 \quad \Rightarrow \quad D = -\frac{1}{\square} G \Lambda^2 h'(GC) . \quad (27)$$

³An alternate choice would have been: $X[g] = G \square^{-1} \square_c^{-1} R \left(\frac{1}{3} R^2 - R_{\mu\nu} R^{\mu\nu} \right)$. It is equally well-motivated and may have interesting comological evolution.

⁴In the models defined by (8) the end of inflation was achieved by the source monotonically increasing without bound leading, unfortunately, to the magnitude problem.

The resulting quantum induced stress tensor should be covariantly conserved:

$$T_{\mu\nu}[g] \equiv \frac{2}{\sqrt{-g}} \frac{\delta(\Delta S)}{\delta g^{\mu\nu}} \quad \Rightarrow \quad D^\mu T_{\mu\nu} = 0 \quad , \quad (28)$$

and a tedious but straightforward computation confirms this.

For the spacetimes of cosmological interest (1), besides (11-13), we have:

$$R_{00} = -3(H^2 + 6\dot{H}) \quad , \quad R_{ij} = (3H^2 + \dot{H})g_{ij} \quad , \quad (29)$$

$$D_0 D_0 = \partial_t^2 \quad , \quad D_i D_j = -g_{ij} H \partial_t \quad . \quad (30)$$

Using all these relations, the specialization to (1) of the equations of motion for the auxiliary scalar fields (21-22) and the Lagrange multipliers (26-27) take the form:

$$\ddot{A} = -3H\dot{A} - (2 - \epsilon)H^2 A - 12(1 - \epsilon)H^4 \quad , \quad (31)$$

$$\ddot{B} = -3H\dot{B} - (2 - \epsilon)H^2 B - 6(2 - \epsilon)H^2 D \quad , \quad (32)$$

$$\ddot{C} = -3H\dot{C} - 6(2 - \epsilon)H^2 A \quad , \quad (33)$$

$$\ddot{D} = -3H\dot{D} + G\Lambda^2 h'(GC) \quad . \quad (34)$$

The full cosmological equations (5) become for the (00) component:

$$\begin{aligned} \frac{3H^2}{16\pi G} + \frac{1}{2}\Lambda^2 h(GC) - \frac{1}{2}(\dot{A}\dot{B} + \dot{C}\dot{D}) - 6H^3\dot{B} \\ - 3(H\partial_t + H^2)\left(\frac{1}{6}AB + AD\right) = \frac{\Lambda}{16\pi G} \quad , \end{aligned} \quad (35)$$

and for the (ij) component:

$$\begin{aligned} - (3 - 2\epsilon)\frac{H^2}{16\pi G} - \frac{1}{2}\Lambda^2 h(GC) + G\Lambda^2 A h'(GC) - \frac{1}{6}\dot{A}\dot{B} - \frac{1}{2}\dot{C}\dot{D} + 2\dot{A}\dot{D} \\ - 2(1 + 2\epsilon)H^3\dot{B} - 2(3 - 2\epsilon)^4(B + 6D) - (H\partial_t + H^2)\left(\frac{1}{6}AB + AD\right) \\ = -\frac{\Lambda}{16\pi G} \quad , \end{aligned} \quad (36)$$

where we have used (31-34) to reach (36). Moreover, it is useful to record the sum (00) + (ij) of these two equations: ⁵

$$\frac{2\epsilon H^2}{16\pi G} + G\Lambda^2 A h'(GC) - \frac{2}{3}\dot{A}\dot{B} - \dot{C}\dot{D} + 2\dot{A}\dot{D} - 4(2 - \epsilon)H^3\dot{B}$$

⁵For instance, one can again check and verify stress-energy conservation: $\partial_t[(00)] = -3H[(00) + (ij)]$.

$$-2(3 - 2\epsilon)H^4(B + 6D) - 4(H\partial_t + H^2)\left(\frac{1}{6}AB + AD\right) = 0 \quad . \quad (37)$$

To describe the evolution of the model it would have been very convenient to use the number of e-foldings n as the time evolution parameter. However, evolution with increasing n is good only as long as the universe expands; if at some point the universe stops expanding or contracts, n cannot describe the evolution because the scale factor $a(n) = \exp(n)$ always increases with increasing n .

We shall therefore use the time variable t as our evolution parameter at the cost of a more complicated system of dynamical equations.⁶ Because these equations will be analyzed numerically it makes sense to use dimensionless variables. If the initial value of the Hubble parameter is $H(t_{\text{in}}) \equiv H_{\text{in}}$, we first define the dimensionless time τ :

$$\tau \equiv H_{\text{in}} t \quad \Rightarrow \quad \partial_t = H_{\text{in}} \partial_\tau \quad , \quad ' \equiv \frac{d}{d\tau} \quad , \quad (38)$$

and then the remaining dimensionless variables:

$$H^2 \equiv \frac{\chi^2}{G} \quad \Rightarrow \quad H_{\text{in}}^2 \equiv \frac{\chi_{\text{in}}^2}{G} \quad , \quad \Lambda \equiv 3H_{\text{in}}^2 \equiv \frac{3\chi_{\text{in}}^2}{G} \quad , \quad (39)$$

$$A \equiv -\frac{3\alpha}{G} \quad , \quad B \equiv -3\beta \quad , \quad C \equiv \frac{9\gamma}{G} \quad , \quad D \equiv \delta \quad , \quad (40)$$

$$h(GC) = h(9\gamma) \equiv f(\gamma) \quad , \quad (41)$$

$$h'(GC) \equiv \frac{\partial}{\partial(GC)} h(GC) = \frac{1}{9} \frac{\partial}{\partial \gamma} f(\gamma) \equiv \frac{1}{9} f'(\gamma) \quad . \quad (42)$$

The set of dimensionless variables we wish to solve for is $\{\alpha, \beta, \gamma, \delta, \chi, \epsilon\}$ and the initial conditions at $\tau = \tau_{\text{in}}$ are:⁷

$$\alpha = \alpha' = \beta = \beta' = \gamma = \gamma' = \delta = \delta' = 0 \quad , \quad (43)$$

$$\chi = \chi_{\text{in}} \quad , \quad \epsilon = 0 \quad . \quad (44)$$

The time evolution of $\{\alpha, \beta, \gamma, \delta\}$ is obtained from the dimensionless form of equations (31-34):

$$\alpha'' + 3\frac{\chi}{\chi_{\text{in}}} \alpha' + (2 - \epsilon)\frac{\chi^2}{\chi_{\text{in}}^2} \alpha = 4(1 - \epsilon)\frac{\chi^4}{\chi_{\text{in}}^2} \quad , \quad (45)$$

⁶Also, to economize on writing we drop the 16π in front of G . Note that there should have been a 16π in the various factors of G that appear in the quantum source $X[g]$ (20).

⁷The initial value data (43) – which follow because $\{\alpha, \beta, \gamma, \delta\}$ all equal to expressions with overall \square^{-1} or \square_c^{-1} in front – ensure that no additional degrees of freedom are introduced by these four fields.

$$\beta'' + 3\frac{\chi}{\chi_{\text{in}}} \beta' + (2 - \epsilon)\frac{\chi^2}{\chi_{\text{in}}^2} \beta = 2(2 - \epsilon)\frac{\chi^2}{\chi_{\text{in}}^2} \delta , \quad (46)$$

$$\gamma'' + 3\frac{\chi}{\chi_{\text{in}}} \gamma' = 2(2 - \epsilon)\frac{\chi^2}{\chi_{\text{in}}^2} \alpha , \quad (47)$$

$$\ddot{\delta} + 3\frac{\chi}{\chi_{\text{in}}} \dot{\delta} = \chi_{\text{in}}^2 f'(\gamma) . \quad (48)$$

Furthermore, the variable χ is solved from the dimensionless form of the (00) equation (35):

$$\begin{aligned} [2\chi_{\text{in}} \beta'] \chi^3 + \left[\frac{1}{3} - \frac{1}{2}\alpha\beta + \alpha\delta \right] \chi^2 + \chi_{\text{in}} \partial_\tau \left[-\frac{1}{2}\alpha\beta + \alpha\delta \right] \chi \\ - \chi_{\text{in}}^2 \left[\frac{1}{3} - \frac{1}{2}\chi_{\text{in}}^2 f(\gamma) + \frac{1}{2}(\alpha'\beta' + \gamma'\delta') \right] = 0 . \end{aligned} \quad (49)$$

This is a cubic algebraic equation which always has a real solution.⁸ The real solution of (49) which is consistent with the correspondence limit of small values for the coefficients of χ^3 and χ is rather complicated:

$$\chi = \frac{1}{3M} \left\{ -1 + \sqrt{1 - 3MN} 2 \cos \left[\frac{\pi}{3} - \frac{1}{3} \arctan Q \right] \right\} , \quad (50)$$

where we have defined:

$$M = \frac{2\chi_{\text{in}} \beta'}{\frac{1}{3} - \frac{\alpha\beta}{2} + \alpha\delta} , \quad (51)$$

$$N = \frac{\chi_{\text{in}} \left(-\frac{\alpha'\beta}{2} - \frac{\alpha\beta'}{2} + \alpha'\delta + \alpha\delta' \right)}{\frac{1}{3} - \frac{\alpha\beta}{2} + \alpha\delta} , \quad (52)$$

$$P = -\frac{\chi_{\text{in}}^2 \left[\frac{1}{3} - \frac{1}{2}\chi_{\text{in}}^2 f(\gamma) + \frac{1}{2}(\alpha'\beta' + \gamma'\delta') \right]}{\frac{1}{3} - \frac{\alpha\beta}{2} + \alpha\delta} , \quad (53)$$

$$Q = \frac{3\sqrt{3}M \sqrt{-P + \frac{N^2}{4} + \frac{9}{2}MNP - MN^3 - \frac{27}{4}M^2P^2}}{1 - \frac{9}{2}MN + \frac{27}{2}M^2P} . \quad (54)$$

Moreover, we solve for ϵ from the dimesionless form of the (00)+(ij) equation (37):

$$\left[2\chi^2 + 12\chi_{\text{in}} \chi^3 \beta' + 12\chi^4 (-\beta + 2\delta) \right] \epsilon = 3 \left\{ \chi_{\text{in}}^4 \alpha f'(\gamma) \right.$$

⁸Had we been able to use n as the evolution parameter, the resulting equation analogous to (49) would have been quadratic in χ^2 .

$$\begin{aligned}
& +\chi_{\text{in}}^2(2\alpha'\beta' + 3\gamma'\delta' + 2\alpha'\delta') - 8\chi_{\text{in}}\chi^3\beta' - 6\chi^4(\beta - 2\delta) \\
& + 4(\chi_{\text{in}}\chi\partial_\tau + \chi^2)\left(\frac{1}{2}\alpha\beta - \alpha\delta\right)\} . \tag{55}
\end{aligned}$$

Finally, it will be useful for the analysis to follow to combine the equations (49, 55) for χ and ϵ and make the simplifications that arise:

$$\begin{aligned}
\epsilon\chi^2 &= \frac{3}{2 + \beta'\chi_{\text{in}}\chi + 12(-\beta + 2\delta)\chi^2} \times \\
&\left\{ [\alpha f'(\gamma) + 2f(\gamma)]\chi_{\text{in}}^4 - \frac{4}{3}\chi_{\text{in}}^2 + (2\alpha' + \gamma')\delta'\chi_{\text{in}}^2 + \frac{4}{3}\chi^2 + 6(-\beta + 2\delta)\chi^4 \right\} . \tag{56}
\end{aligned}$$

Finally, we must make a choice for the function $f(X)$. The perturbative result (4) indicates that the effect gets strong when $G\Lambda H_{\text{in}}t \sim 1$, and this corresponds to $X \sim 1$. A simple appropriate singular algebraic function is:

$$f(X[g]) = \frac{X[g]}{1 - X[g]} = \frac{1}{1 - X[g]} - 1 , \tag{57}$$

which also has the property that the small X limit of $f(X)$ is X .

3 The Resulting Cosmology

The purpose of this section is to describe the sort of background cosmology this model produces. We begin with a discussion of inflation and how it ends, then we describe the immediate post-inflationary phase. These portions of the treatment are supported by substantial numerical analysis, reported in the form of graphs. Subsequent evolution involves matter in an essential way, so we limit the discussion to some general comments.

3.1 The Inflationary Regime

For a long period the scale factor remains at nearly its de Sitter value of $a(\tau) = e^\tau$. During this phase the four scalars experience some minor transients which decay like powers of $e^{-\tau}$ to reveal forms which persist until screening becomes significant:

$$\alpha(\tau) = 2\chi_{\text{in}}^2(1 - e^{-\tau})^2 \longrightarrow 2\chi_{\text{in}}^2 , \tag{58}$$

$$\beta(\tau) = \frac{2}{3}\chi_{\text{in}}^2\left(\tau - \frac{11}{6} + 3e^{-\tau} - \frac{3}{2}e^{-2\tau} + \frac{1}{3}e^{-3\tau}\right) \longrightarrow \frac{2}{3}\chi_{\text{in}}^2\left(\tau - \frac{11}{6}\right) , \tag{59}$$

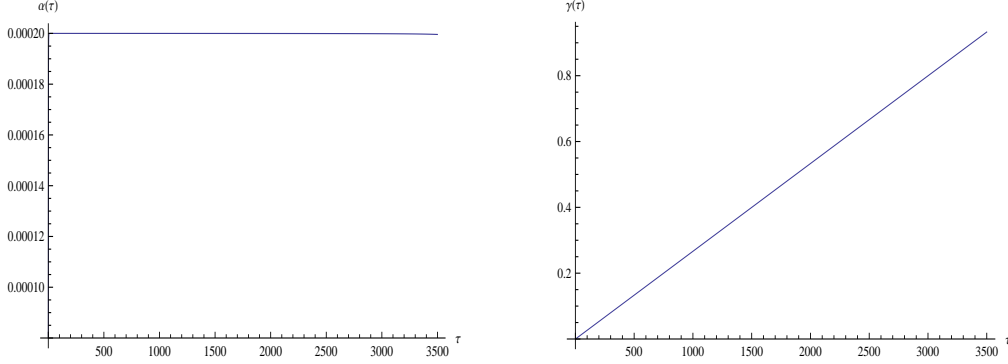


Figure 1: Numerical simulation of the auxiliary scalar functions $\alpha(\tau)$ and $\gamma(\tau)$ for $\chi_{\text{in}} = \frac{1}{100}$ and $f(X) = \frac{X}{1-X}$. Note the quantitative agreement with expressions (58) and (60) during the full epoch of de Sitter expansion.

$$\gamma(\tau) = \frac{8}{3}\chi_{\text{in}}^2\left(\tau - \frac{11}{6} + 3e^{-\tau} - \frac{3}{2}e^{-2\tau} + \frac{1}{3}e^{-3\tau}\right) \longrightarrow \frac{8}{3}\chi_{\text{in}}^2\left(\tau - \frac{11}{6}\right), \quad (60)$$

$$\delta(\tau) = \frac{1}{3}\chi_{\text{in}}^2\left(\tau - \frac{1}{3} + \frac{1}{3}e^{-3\tau}\right) \longrightarrow \frac{1}{3}\chi_{\text{in}}^2\left(\tau - \frac{1}{3}\right). \quad (61)$$

These behaviours are evident in Figures 1 and 2, which were generated for $\chi_{\text{in}} = \frac{1}{100}$ and $f(X) = \frac{X}{1-X}$.

From relations (58-61) we see that derivatives during the de Sitter epoch take the form:

$$\alpha'(\tau) \longrightarrow 0, \quad \beta'(\tau) \longrightarrow \frac{2}{3}\chi_{\text{in}}^2, \quad \gamma'(\tau) \longrightarrow \frac{8}{3}\chi_{\text{in}}^2, \quad \delta'(\tau) \longrightarrow \frac{1}{3}\chi_{\text{in}}^2. \quad (62)$$

We can also see $-\frac{1}{2}\beta(\tau) + \delta(\tau) \rightarrow \frac{1}{2}\chi_{\text{in}}^2$. Using these relations in the expressions for the Hubble parameter and the first slow roll parameter imply:

$$\chi(\tau) \longrightarrow \chi_{\text{in}}\left(1 - 2\chi_{\text{in}}^4\tau\right), \quad \epsilon(\tau) \longrightarrow 2\chi_{\text{in}}^4. \quad (63)$$

Figure 3 shows that expressions (63) are in rough agreement with numerical simulation.

Figure 1 extends to $\tau = 3500$, and shows essentially perfect agreement with expressions (58) and (60). However, Figure 2 extends only to $\tau = 1000$, and shows a small curvature in addition to the linear behaviour predicted by expressions (59) and (61). This curvature becomes more pronounced for larger values of τ , as is evident in Figure 4. The curvature derives from two couplings between the auxiliary scalars which are small but not negligible

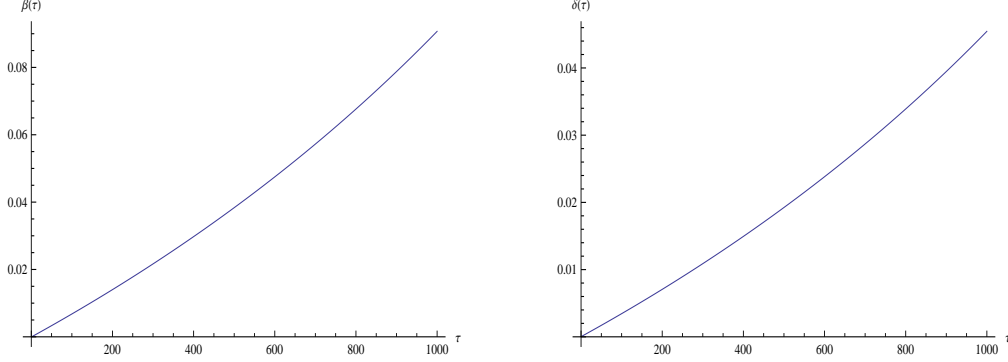


Figure 2: Numerical simulation of the auxiliary scalar functions $\beta(\tau)$ and $\delta(\tau)$ for $\chi_{\text{in}} = \frac{1}{100}$ and $f(X) = \frac{X}{1-X}$. Note the rough agreement with expressions (59) and (61) during the early epoch of de Sitter expansion.

during the de Sitter epoch. The first is of $\delta(\tau)$ to $\gamma(\tau)$ (we are assuming $f(X) = \frac{X}{1-X}$):

$$\delta'' = -3\frac{\chi}{\chi_{\text{in}}} \delta' + \chi_{\text{in}}^2 f(\gamma) \longrightarrow -3\delta' + \chi_{\text{in}}^2 (1+2\gamma) . \quad (64)$$

The order χ_{in}^4 correction to $\delta(\tau)$ comes from the linear growth of $\gamma(\tau)$ in (60):

$$\delta(\tau) \longrightarrow \frac{1}{3}\chi_{\text{in}}^2 \tau + \frac{8}{9}\chi_{\text{in}}^4 \tau^2 . \quad (65)$$

The curvature of $\beta(\tau)$ descends from this growth, as reflected in the coupling between β and δ :

$$\beta'' = -3\frac{\chi}{\chi_{\text{in}}} \beta' - (2-\epsilon)\frac{\chi^2}{\chi_{\text{in}}^2} \beta + 2(2-\epsilon)\frac{\chi^2}{\chi_{\text{in}}^2} \delta \longrightarrow -3\beta' - 2\beta + 4\delta . \quad (66)$$

It follows that the order χ_{in}^4 correction to $\beta(\tau)$ is:

$$\beta(\tau) \longrightarrow \frac{2}{3}\chi_{\text{in}}^2 \tau + \frac{16}{9}\chi_{\text{in}}^4 \tau^2 . \quad (67)$$

The first effect of the curvature of $\beta(\tau)$ and $\delta(\tau)$ is to make $\epsilon(\tau)$ grow slightly. That can be seen in the right hand graph of Figure 3. Curvature also causes the Hubble parameter to decline faster than linearly, as can be seen in the left hand graph of Figure 5. From expression (60), and the fact that $f(X) = \frac{X}{1-X}$ becomes singular at $X = 1$, one can estimate that inflation comes to an end at about $\tau \simeq \frac{3}{8}\chi_{\text{in}}^{-2} = 3750$. The right hand graph of Figure 5 reveals that the actual point where $\epsilon = 1$ is about $\tau \simeq 3757.3$. Figure 6 shows the Hubble parameter during this period.

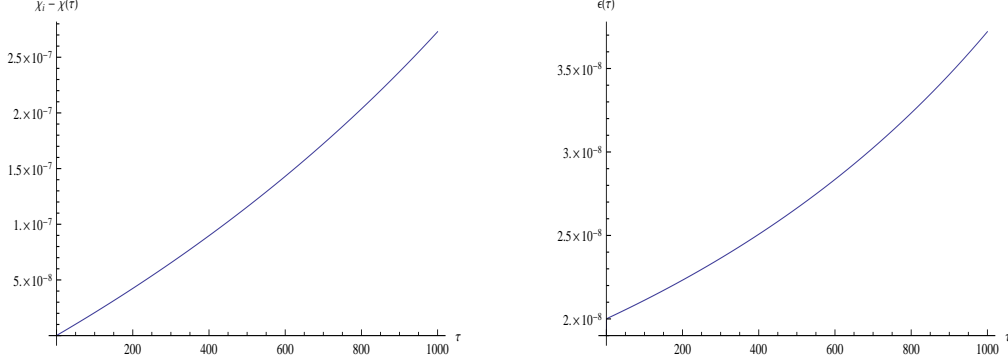


Figure 3: Numerical simulation of the geometrical quantities $\chi_{\text{in}} - \chi(\tau)$ and $\epsilon(\tau)$ for $\chi_{\text{in}} = \frac{1}{100}$ and $f(X) = \frac{X}{1-X}$. Note the rough agreement with (3) during the early epoch of nearly de Sitter expansion.

3.2 Reheating and Radiation Domination

Figure 6 shows that screening becomes effective quite suddenly and brings inflation to an end at about $\tau \simeq 3757.3$. The first slow roll parameter goes from $\epsilon = 0.3$ to $\epsilon = 1$ over a period of only $\Delta\tau \simeq 2$. Thereafter, we see from Figure 7 that the Hubble parameter oscillates with a decreasing amplitude. Of course $|\chi(\tau)| \gg \chi_{\text{in}}$, and Hubble friction ceases to be effective. Close examination of Figure 7 also reveals that the magnitude of $\chi' = -\epsilon\chi^2$ is about 50 times larger than χ^2 . All of this justifies simplifying the auxiliary scalar equations accordingly:

$$\alpha'' \simeq +\frac{\epsilon\chi^2}{\chi_{\text{in}}^2}\alpha \quad , \quad \gamma'' \simeq -\frac{2\epsilon\chi^2}{\chi_{\text{in}}^2}\alpha \quad , \quad (68)$$

$$\delta'' \simeq \chi_{\text{in}}^2 f'(\gamma) \quad , \quad \beta'' \simeq \frac{\epsilon\chi^2}{\chi_{\text{in}}^2}(\beta - 2\delta) \quad . \quad (69)$$

Figure 8 shows that $\alpha(\tau)$ and $\gamma(\tau)$ experience oscillations of decreasing amplitude about central values of $\alpha_0 \simeq 0.0001015$ and $\gamma_0 \simeq 0.99981$, respectively. This means one can carry the simplifications of the equations (68) a step further:

$$\left(\alpha'' \simeq \alpha_0 \frac{\epsilon\chi^2}{\chi_{\text{in}}^2} \quad \& \quad \gamma'' \simeq -2\alpha_0 \frac{\epsilon\chi^2}{\chi_{\text{in}}^2} \right) \implies \Delta\alpha(\tau) \equiv \alpha(\tau) - \alpha_0 \simeq -\frac{1}{2}[\gamma(\tau) - \gamma_0] \equiv -\frac{1}{2}\Delta\gamma(\tau) \quad . \quad (70)$$

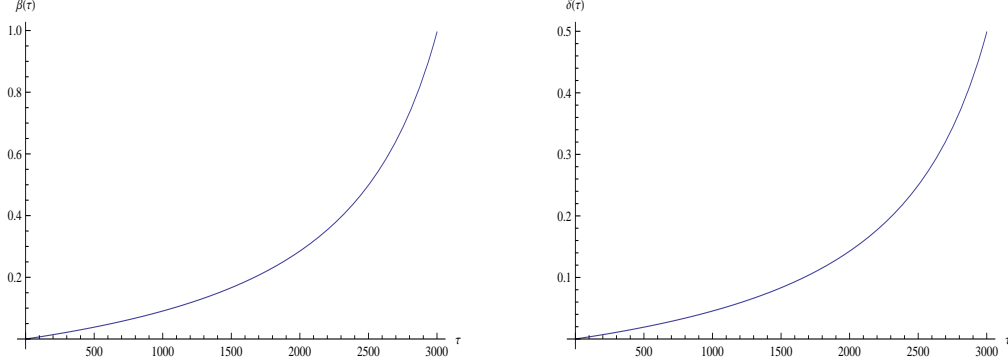


Figure 4: Numerical simulation of the auxiliary scalar functions $\beta(\tau)$ and $\delta(\tau)$ for $\chi_{\text{in}} = \frac{1}{100}$ and $f(X) = \frac{X}{1-X}$. Note the curvature quantitative agreement with expressions (59) and (61) during the early epoch of de Sitter expansion.

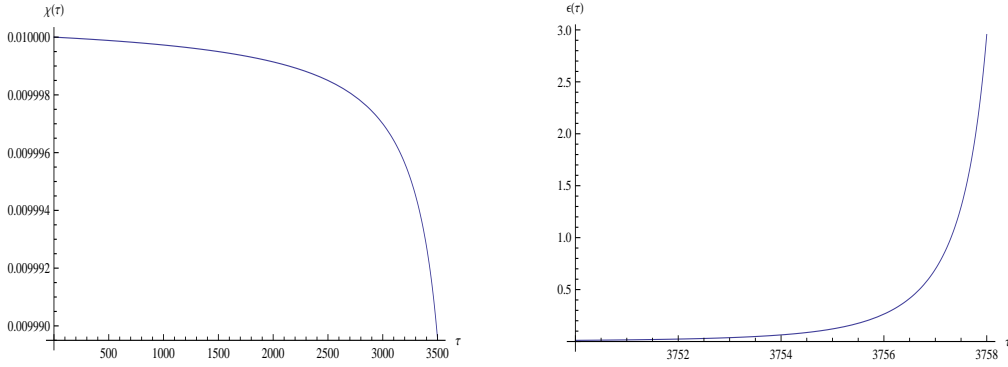


Figure 5: Numerical simulation of the Hubble parameter and the first slow roll parameter for $\chi_{\text{in}} = \frac{1}{100}$ and $f(X) = \frac{X}{1-X}$.

At this stage we can gain a rough understanding of what is driving the oscillations. Recall expression (56) for $\epsilon\chi^2$ and use $\chi^2 \ll \chi_{\text{in}}^2$ as well as $2\alpha' + \gamma' \simeq 0$ to simplify the numerator of (56):

$$\begin{aligned} & \left[\alpha f'(\gamma) + 2f(\gamma) \right] \chi_{\text{in}}^4 - \frac{4}{3} \chi_{\text{in}}^2 + (2\alpha' + \gamma') \delta' \chi_{\text{in}}^2 + \frac{4}{3} \chi^2 + 6(-\beta + 2\delta) \chi^4 \\ & \simeq \left[\alpha f'(\gamma) + 2f(\gamma) \right] \chi_{\text{in}}^4 - \frac{4}{3} \chi_{\text{in}}^2, \end{aligned} \quad (71)$$

$$\simeq \left[\alpha_0 f'(\gamma_0) + 2f(\gamma_0) \right] \chi_{\text{in}}^4 - \frac{4}{3} \chi_{\text{in}}^2 + \left[\frac{3}{2} f'(\gamma_0) + \alpha_0 f''(\gamma_0) \right] \chi_{\text{in}}^4 \times \Delta\gamma, \quad (72)$$

$$\simeq 0.711 \times \Delta\gamma. \quad (73)$$

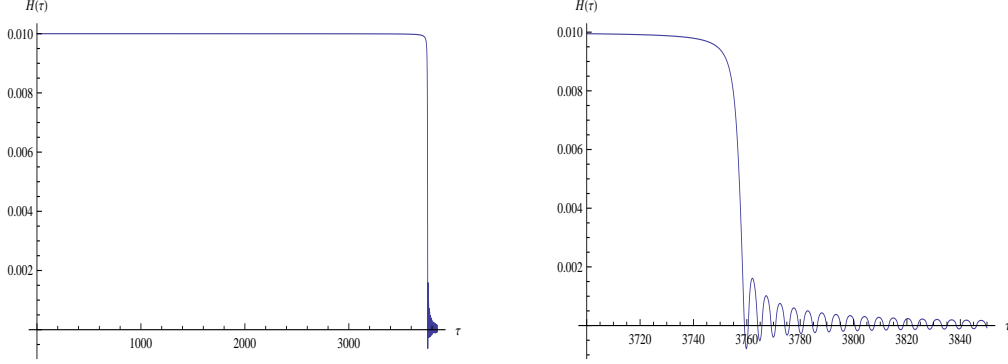


Figure 6: Numerical simulation of the Hubble parameter for $\chi_{\text{in}} = \frac{1}{100}$ and $f(X) = \frac{X}{1-X}$.

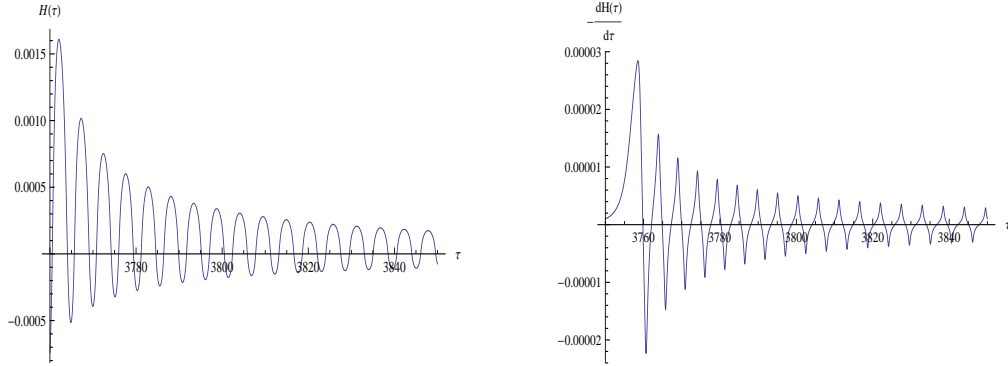


Figure 7: Numerical simulation for $\chi_{\text{in}} = \frac{1}{100}$ and $f(X) = \frac{X}{1-X}$ of the Hubble parameter and its first time derivative in the period after the end of inflation.

Substituting (73) into (56), and then into (70) gives what is a recognizable oscillator equation for $\Delta\gamma(\tau)$:

$$\Delta\gamma'' \simeq -\frac{4.333 \times \Delta\gamma}{2 + 12\beta' \chi_{\text{in}} \chi + 12(-\beta + 2\delta)\chi^2} . \quad (74)$$

Of course this also implies oscillations for $\Delta\alpha \simeq -\frac{1}{2}\Delta\gamma$, and for $\epsilon\chi^2$. The decreasing amplitude of oscillation is presumably due to the residual effect of Hubble friction.

Figure 9 shows the auxiliary scalars $\delta(\tau)$ and $\beta(\tau)$. We can understand the growth of $\delta(\tau)$ by making a further simplification of its equation (69):

$$\delta'' \simeq \chi_{\text{in}}^2 f'(\gamma_0) \simeq 2770 . \quad (75)$$

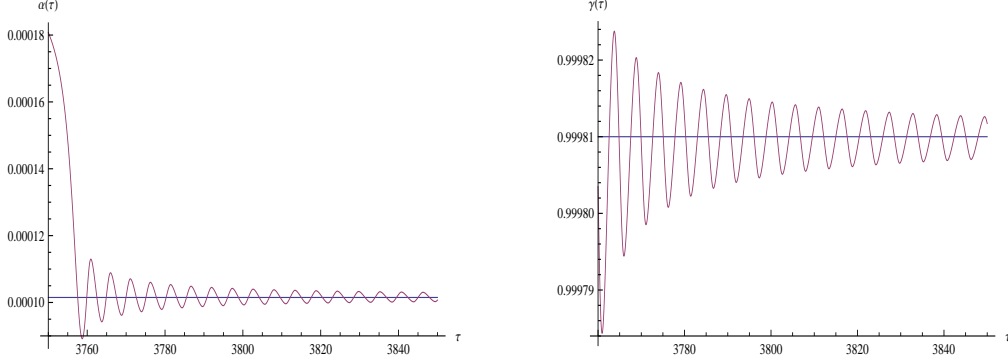


Figure 8: Numerical simulation of auxiliary scalars $\alpha(\tau)$ and $\gamma(\tau)$ after the end of inflation for $\chi_{\text{in}} = \frac{1}{100}$ and $f(X) = \frac{X}{1-X}$. The scalar $\alpha(\tau)$ (on the left) oscillates around $\alpha_0 \simeq 0.0001015$. The scalar $\gamma(\tau)$ (on the right) oscillates around $\gamma_0 \simeq 0.99981$.

That would give quadratic growth. What Figure 9 actually shows is somewhat slower growth, $\delta(\tau) \simeq \frac{2770}{2} \times (\tau - 2760)^{1.82}$. The reduction seems to be due to the residual effect of Hubble friction. We can understand the behaviour of $\beta(\tau)$ by making a similar simplification of its equation (69):

$$\beta'' \simeq -\frac{2\epsilon\chi^2}{\chi_{\text{in}}^2} \delta. \quad (76)$$

The source on the right hand side oscillates and grows linearly, so the response of $\beta(\tau)$ in Figure 9 can be understood by stripping away all the constants:

$$\left\{ \begin{array}{l} f''(x) = -x \cos(x) \\ f(0) = 0 = f'(0) \end{array} \right\} \implies f(x) = x [\cos(x) + 1] - 2 \sin(x). \quad (77)$$

The preceding analysis and numerical results have dealt with the period immediately after the end of inflation. A point of great significance is that *the Hubble parameter becomes negative*. This is evident in Figure 7. Of course negative H means that the universe is contracting, which must concentrate whatever matter particles are produced by the fluctuating geometry. There will be a similar concentration of the last graviton and inflaton perturbations to have been generated during inflation. We are not now in a position to analyze this process in detail but it seems obvious that rapid reheating will occur. And note that the onset of radiation domination, with $\epsilon = 2$, turns off the source for further evolution of the key auxiliary scalar $\gamma(\tau)$. Assuming that this point is reached, the screening effect goes quiescent with $\gamma = \gamma_*$,

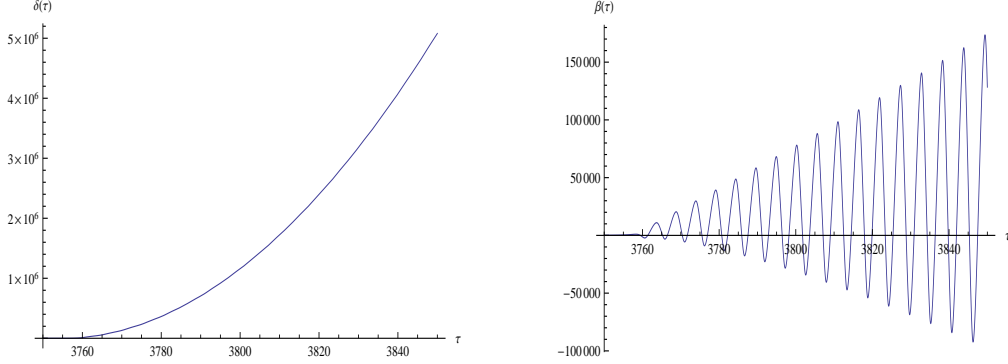


Figure 9: Numerical simulation of auxiliary scalars $\delta(\tau)$ and $\beta(\tau)$ after the end of inflation for $\chi_{\text{in}} = \frac{1}{100}$ and $f(X) = \frac{X}{1-X}$. The scalar $\delta(\tau)$ (on the left) grows, roughly like $(\tau - 2760)^{1.82}$. The scalar $\beta(\tau)$ (on the right) oscillates with a linearly increasing central value and a linearly increasing amplitude.

and the universe experiences the usual phase of radiation domination with essentially zero cosmological constant.

3.3 Late Time Acceleration

We turn now to the time $t = t_m$, long after reheating, when the universe makes the transition to matter domination. At that point γ begins to evolve again. This will induce a corresponding change in our non-local source (10):

$$\begin{aligned} \Lambda^2 h(GC) &= \Lambda^2 f(\gamma) = \Lambda^2 f\left(G \frac{1}{9} \frac{1}{\square} R \frac{1}{\square_c} \left[\frac{1}{3} R^2 - R_{\mu\nu} R^{\mu\nu}\right]\right) \\ &\simeq \Lambda^2 f(\gamma_*) + \Lambda^2 f'(\gamma_*) \times [\gamma - \gamma_*] . \end{aligned} \quad (78)$$

Let us first examine the change in $[\gamma - \gamma_*](t)$, returning to co-moving time as an evolution variable:

$$\begin{aligned} [\gamma - \gamma_*](t) &= \frac{G}{9} \int_{t_m}^t \frac{dt'}{a^3(t')} \int_{t_m}^{t'} dt'' a^3(t'') \times 6[2 - \epsilon(t'')] H^2(t'') \\ &\quad \times \frac{1}{a(t'')} \int_{t_m}^{t''} \frac{dt'''}{a(t''')} \int_{t_m}^{t'''} dt'''' a^2(t''') \times 12[1 - \epsilon] H^4(t''') . \end{aligned} \quad (79)$$

To simplify the discussion, we assume that the universe is perfectly matter dominated after $t = t_m$:

$$t > t_m \quad \implies \quad a(t) = a_m \left(\frac{t}{t_m}\right)^{\frac{2}{3}} , \quad H(t) = \frac{2}{3t} , \quad \epsilon(t) = \frac{3}{2} . \quad (80)$$

Substituting (80) into (79) and performing the trivial integrations gives:

$$[\gamma - \gamma_*](t) = -\frac{2^7 G}{3^3 5 t_m^2} \left\{ \frac{5}{6} - 3 \left(\frac{t}{t_m} \right)^{\frac{1}{3}} + \frac{15}{4} \left(\frac{t_m}{t} \right)^{\frac{2}{3}} - \frac{5}{3} \frac{t_m}{t} + \frac{1}{12} \left(\frac{t_m}{t} \right)^2 \right\}. \quad (81)$$

It is more useful to express this in terms of the Hubble parameter at the time of matter domination $H_m \equiv H(t_m)$, and to take the late time limit:

$$\lim_{t \gg t_m} [\gamma - \gamma_*](t) = -\frac{16}{9} G H_m^2. \quad (82)$$

Let us now turn to the question of what sort of function $f(X)$ would give a late time cosmological constant of the right size. The two terms on the right hand side of (78) have different roles:

- The first cancels the large, bare cosmological constant:

$$\frac{6 H_{\text{in}}^2}{16 \pi G} = \Lambda^2 f(\gamma_*) ; \quad (83)$$

- The second supplies the small, positive cosmological constant needed to cause the observed late time acceleration:

$$\frac{6 H_0^2}{16 \pi G} = \Lambda^2 f'(\gamma_*) \times \frac{16}{9} G H_m^2. \quad (84)$$

Taking the ratio of (84) to (83) implies $f(X)$ must obey:

$$\frac{f'(\gamma_*)}{f(\gamma_*)} = \frac{9}{16} \left(\frac{H_0}{H_{\text{in}}} \right)^2 \frac{1}{G H_m^2}. \quad (85)$$

Some of the numbers in expression (85) are known:

$$G H_m^2 \simeq 10^{10} \times G H_0^2 \simeq 10^{-112}. \quad (86)$$

If we assume $H_{\text{in}} \simeq 10^{55} H_0$, the result is:

$$\frac{f'(\gamma_*)}{f(\gamma_*)} \simeq 10^2. \quad (87)$$

Conditions (83) and (87) are certainly not obeyed for the simple ansatz $f(X) = \frac{X}{1-X}$ that was used for our numerical simulations. We therefore consider a 1-parameter family of more singular models:

$$f(X) = \frac{1}{(1-X)^\omega} - 1 \quad \implies \quad f'(X) = \frac{\omega}{(1-X)^{\omega+1}}. \quad (88)$$

Because γ_* is very close to unity we can express conditions (83) and (87) as:

$$\frac{1}{(1-\gamma_*)^\omega} \simeq 10^{10} \quad , \quad \frac{\omega}{1-\gamma_*} \simeq 10^2 . \quad (89)$$

An approximate solution is clearly $\omega \simeq 5$. Different assumptions about H_i can be accommodated with only small changes in the exponent ω . Note also that because the actual expansion history is not (80), the current phase of acceleration will eventually end, although after a very long time.

4 Epilogue

The model presented in this paper was ultimately motivated by the fact that gravitation is the dominant force responsible for the evolution of the universe. It is therefore reasonable to seek a model exclusively using gravitational degrees of freedom. Its construction was dictated by consistency with perturbative results as well as with satisfaction of basic cosmological requirements. Alternatively, its construction can be viewed as simply an *ansatz* whose implications should be studied.

These implications include an end by gravitational means of an inflationary era of adequate duration, an oscillatory era that follows and can lead a naturally reheated universe to the epoch of radiation domination and, thereafter, to matter domination. The analysis is based on numerical and semi-analytical methods.

Finally, the several conjectures we have had to make in this analysis should not be allowed to obscure the fact that this model provides natural explanations both for why the current phase of acceleration happens so late in cosmological history, and for why its source appears to be an absurdly small, positive cosmological constant. Our key point is that late time acceleration is not the result of a small bare cosmological constant but rather of a very small fractional change in quantum gravitational screening which was triggered by the transition from radiation domination to matter domination.

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